Garbage disposal game on finite graphs

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Abstract

The garbage disposal game involves a finite set of individuals, each of whom updates their garbage by either receiving from or dumping onto others. Instead of applying game theory, we investigate the game from a mathematical perspective. We examine the case where only social neighbors, whose garbage levels differ by at most a given threshold, can offload an equal proportion of their garbage onto others. Remarkably, in the absence of this threshold, the garbage amounts of all individuals converge to the initial average on any connected social graph of order at least 3.

1 Introduction

The garbage disposal game comprises a set of n individuals. Each individual updates their garbage either by receiving garbage from others or by dumping garbage onto others [8]. Mathematically, let $[n]=\{1,\ldots,n\}$ represent the set of all individuals, and let $x_i(t)\geq 0$ denote the amount of garbage held by individual i at time t. The update rule for individual i's garbage is given by $x_i(t+1)=\sum_{j\in[n]}A_{ij}(t)x_j(t)$, where $A_{ij}(t)\in[0,1]$ represents the proportion of individual j's garbage that is dumped onto individual i at time t. This ensures that $\sum_{i\in[n]}A_{ij}(t)=1$. A vector is *stochastic* if all entries are nonnegative and add up to 1. A square matrix is *row-stochastic* if each row is stochastic, and *column-stochastic* if each column is stochastic. Writing the update mechanism in matrix form:

$$x(t+1)=A(t)x(t) \tag{1}$$

where

$$x(t)=\text{transpose of }(x_1(t),\ldots,x_n(t))=(x_1(t),\ldots,x_n(t))'\in\mathbb{R}^n_{\geq 0},$$
$$A(t)\in\mathbb{R}^{n\times n}\text{ is column-stochastic with the }(i,j)\text{-th entry }A_{ij}(t).$$

The utility of individual i at time t is $u_i(x_i(t))$, where u_i is a decreasing function. This indicates that the more garbage an individual processes, the less utility they derive.

Unlike certain opinion models, such as the voter model, the threshold voter model and the asynchronous Hegselmann-Krause model, where an agent solely updates their opinion at each time step, an agent in the garbage disposal game cannot update their garbage independently if they dump onto others [14, 3, 13, 11, 5, 1, 6, 7, 16, 4, 9]. In the Hegselmann-Krause (HK) model, an agent updates their opinion by averaging the opinions of their opinion neighbors. In the synchronous HK model, all agents update their opinions at each time step, whereas in the asynchronous HK model, only one agent, uniformly selected at random, updates their opinion at each time step [12, 13, 15]. The HK model belongs to averaging dynamics but is not necessarily

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a garbage disposal game. Moreover, the matrix $A(t)$ is not always row-stochastic, so the garbage disposal game does not necessarily belong to averaging dynamics.

In this paper, we consider the garbage disposal game, in which an individual can dump garbage onto others if and only if they are social neighbors and their garbage differs by at most a confidence threshold $\epsilon > 0$, which is a random variable. In detail, let $G = ([n], E)$ be an undirected simple social graph with vertex set $[n]$ and edge set E. An edge $(i, j) \in E$ symbolizes that agents i and j are social neighbors. Let $G_t = ([n], E_t)$ be a subgraph of G at time t with vertex set $[n]$ and edge set $E_t = \{(i, j) \in E : |x_i(t) - x_j(t)| \leq \epsilon\}$, recording agents who are social neighbors and whose garbage differs by at most the threshold ϵ. $N_i(t) = \{j \in [n] : (i, j) \in E_t\}$ includes all social neighbors of agent i at time t whose garbage differs by at most the threshold ϵ. The proportion of agent j's garbage dumping onto agent i equals $\frac{1}{|E_t|}$ if $(i, j) \in E_t$, $1 - \frac{|N_i(t)|}{|E_t|}\mathbb{1}\{E_t \neq \emptyset\}$ if $i = j$, and 0 otherwise. Namely,

$$A_{ij}(t) = \frac{1}{|E_t|}\mathbb{1}\{(i, j) \in E_t\} \quad \text{if } i \neq j, \quad A_{ii}(t) = 1 - \frac{|N_i(t)|}{|E_t|}\mathbb{1}\{E_t \neq \emptyset\}.$$

Thus, agents i and j dump the same proportion of their garbage onto each other if they are social neighbors and their garbage differs by at most the threshold ϵ. For instance, if G_t is the star graph of order 6, as shown in Figure 1, then agent 1 can dump $\frac{1}{5}$ of their garbage onto each of the other agents, and each of the other agents can dump $\frac{1}{5}$ of their garbage onto agent 1. Note that agent 1 is the center of the star graph and is the only agent emptying their original garbage. The update mechanism is as follows for all $i \in [n]$:

$$x_i(t+1) = \frac{\mathbb{1}\{E_t \neq \emptyset\}}{|E_t|} \sum_{j \in N_i(t)} x_j(t) + \left(1 - \frac{|N_i(t)|}{|E_t|}\mathbb{1}\{E_t \neq \emptyset\}\right) x_i(t). \tag{2}$$

Observe that the garbage disposal game described in (2) falls within the category of averaging dynamics. We assume that $x_i(0)$, $i \in [n]$, are nonnegative real-valued random variables. Let $a \wedge b$ and $a \vee b$ denote the *minimum* and *maximum* of a and b, respectively. A graph G is termed δ-*trivial* if the distance between any two vertices is at most δ. Let $V(H)$ represent the vertex set of the graph H. Let $\mathbb{1}$ denote the vector with all entries equal to 1. The *Laplacian* $\mathscr{L}$ on the simple graph $G = ([n], E)$ is defined as $\mathscr{L} = \text{diag}(d_1, \ldots, d_n) - A$, where d_i is the degree of vertex i in G, and A is the adjacency matrix of G with $A_{ij} = \mathbb{1}\{(i, j) \in E\}$. The *convex hull* generated by $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ is the smallest convex set containing $v_1, v_2, \ldots, v_n$. It is defined as follows:

$$C(\{v_1, v_2, \ldots, v_n\}) = \left\{v : v = \sum_{i=1}^{n} \lambda_i v_i \text{ where } (\lambda_i)_{i=1}^{n} \text{ is a stochastic vector}\right\}.$$

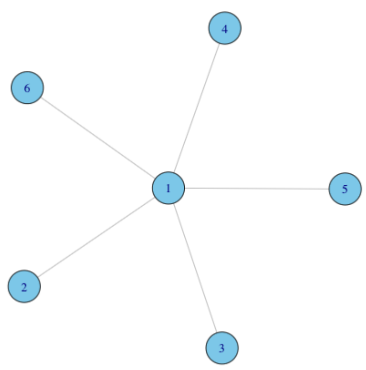

Figure 1: Star graph of order 6

2 Main Results

The ϵ-triviality of the graph G_s implies $G = G_s$, which is equivalent to the case without the threshold ϵ. We will later prove that the graph G_t preserves δ-triviality over time for all $\delta > 0$, leading to $G_t = G$ for all $t \geq s$. Thus, from time s onward, the garbage disposal game is played on the social graph G. Consequently, Theorem 1 shows that, without the threshold ϵ, the garbage amounts of all agents will converge to the initial average on any connected social graph of order at least 3. Note that setting $\epsilon = \max_{i,j\in[n]} |x_i(0) - x_j(0)|$ is equivalent to having no threshold ϵ.

Theorem 1. *Assume that*

- *the social graph G is connected with $n \geq 3$, and*
- *the graph G_s is ϵ-trivial at some time $s \geq 0$.*

Then, $x_i(t)$ converges to $\frac{1}{n}\sum_{k\in[n]} x_k(0)$ as $t \to \infty$ for all $i \in [n]$.

Observe that when $n = 2$, it is a process of mutually interchanging garbage. Under the assumption of Theorem 1, those whose garbage is above average benefit from the game, while those whose garbage is below average suffer in the game.

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